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Convergence Properties of Sequences of Functions with Application to Restricted Derivative Approximation

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Convergence properties of sequences of continuous functions, with kth order divided differences bounded from above or below, are studied. It is found that for such sequences, convergence in a "monotone norm" (e.g., L_p) on $[a, b]$ to a continuous function implies uniform convergence of the sequence and its derivatives up to order $k - 1$ (whenever they exist), in any closed subinterval of $[a, b]$. Uniform convergence in the closed interval $[a, b]$ follows from the boundedness from below and above of the kth order divided differences. These results are applied to the estimation of the degree of approximation in Monotone and Restricted Derivative approximation, via bounds for the same problems with only one restricted derivative.

INTRODUCTION

In a study of "Restricted Derivative Approximation" (R.D.A.) to functions with one derivative outside the range [7], the impossibility of approximating such functions arbitrarily closely was proved. These results lead to the observation that uniform convergence of a sequence of functions with restricted kth derivative implies the uniform convergence of the sequences of derivatives up to order $k - 1$. Motivated by this idea and the results of [9], we were able to extend and generalize the above results to sequences of functions with kth order divided differences bounded from below or above, which converge to a continuous function in a "monotone norm" (i.e., a norm with the property $|f(x)| \leqslant |g(x)|,\ a \leqslant x \leqslant b \Rightarrow \|f\| \leqslant \|g\|$). It is found that such sequences converge uniformly on any closed subinterval of $[a, b]$. This property is also shared by the sequences of derivatives up to order $k - 1$, whenever they exist. By investigating the behavior of the sequence at a and b, we found sufficient conditions for uniform convergence in the closed interval $[a, b]$.

The present approach does not use the results in [7] but is direct, with calculations similar to those in [9]. Nevertheless the results have significant

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implications to the problems of R.D.A. and Monotone Approximation (M.A.). It is shown that for certain functions the degree of approximation in R.D.A. and M.A. can be estimated by their degree of approximation from similar classes but with only one restricted derivative.

It should be noted that a special case of one of our main results, i.e., that pointwise convergence of distribution functions implies their uniform convergence, is known in a probability context [5, p. 268]. It seems plausible that the results of this work have further implications in this direction.

In Section 1 we present some properties of divided differences and some useful notations and definitions. Section 2 includes the main results about sequences of functions and their convergence properties, and the applications to R.D.A. and M.A. are given in Section 3.

1. Preliminaries and Notation

In this section we present some properties of divided differences which are easy to verify [13], and define some concepts to be used in subsequent sections.

Let $f[x_0, x_1, ..., x_k]$ be the kth divided difference of $f(x)$ at $k+1$ distinct points, given by:

$$f[x_0, x_1, ..., x_k] = \sum_{\nu=0}^{k} \frac{f(x_\nu)}{w'(x_\nu)}, \tag{1.1}$$

where $w(x) = \prod_{\mu=0}^{k} (x - x_\mu)$. If $f(x)$ has a kth derivative at a point x_0 then

$$f^{(k)}(x_0) = \lim_{h\to 0} k! f[x_0, x_0 + h, ..., x_0 + kh]. \tag{1.2}$$

In the sequel we deal with functions which satisfy a restriction of the form

$$\sigma f[x_0, ..., x_k] \geqslant M \qquad \text{for all } a \leqslant x_0 < x_1 < \cdots < x_k \leqslant b, \tag{1.3}$$

where $\sigma = +1$ or -1, and $-\infty < M < \infty$ is a constant. For $M = 0$ such functions are called k-convex and for $k \geqslant 2$ belong to $C^{k-2}(a, b)$ [1]. Since (1.3) is equivalent to

$$\left(\sigma f - \frac{Mx^k}{k!}\right)[x_0, x_1, ..., x_k] \geqslant 0, \tag{1.4}$$

any function satisfying (1.3) for $k \geqslant 2$ belongs to $C^{k-2}(a, b)$. The following lemma can be derived from (1.4) and the results in [1] on k-convex functions. Yet, in order to avoid details and notation not relevant to this work, we present a direct proof:

LEMMA 1.1. *Let* $f(x)$ *satisfy* (1.3) *for some* $k \geqslant 1$. *Then for any* $1 \leqslant j \leqslant k$ *such that* $f^{(j)}(x)$ *exists in* $[a, b]$

$$\sigma f^{(j)}[x_0, x_1, \ldots, x_{k-j}] \geqslant \frac{k!}{(k-j)!} M$$

for all $a \leqslant x_0 < x_1 < \cdots < x_{k-j} \leqslant b$.

Proof. Obviously, if $g[x_0, x_1] \geqslant M$ for all $a \leqslant x_0 < x_1 \leqslant b$ and $g(x)$ is differentiable in $[a, b]$, then $g'(x) \geqslant M$ for all $x \in [a, b]$. Since the kth divided difference is the first divided difference of the $(k-1)$th, (1.3) implies $(\partial/\partial x_i)\, \sigma f[x_0, x_1, \ldots, x_{k-1}] \geqslant M$ for all distinct $x_0, x_1, \ldots, x_{k-1}$ in $[a, b]$ and $i = 0, 1, \ldots, k-1$. Using the identity

$$\sum_{i=0}^{k-1} \frac{\partial}{\partial x_i} F(x_0 - x_1, x_0 - x_2, \ldots, x_0 - x_{k-1}) = 0$$

and others like it, we get

$$\begin{aligned} &\sum_{i=0}^{k-1} \frac{\partial}{\partial x_i} \sigma f[x_0, x_1, \ldots, x_{k-1}] \\ &\quad = \sigma \sum_{i=0}^{k-1} \frac{\partial}{\partial x_i} \sum_{j=0}^{k-1} \frac{f(x_j)}{\omega'(x_j)} = \sigma \left\{ \sum_{i=0}^{k-1} \frac{f'(x_i)}{\omega'(x_i)} \right. \\ &\quad\quad \left. + \sum_{j=0}^{k-1} f(x_j) \sum_{i=0}^{k-1} \frac{\partial}{\partial x_i} \frac{1}{\omega'(x_j)} \right\} = \sigma f'[x_0, x_1, \ldots, x_{k-1}], \end{aligned}$$

where $w(x) = \prod_{\mu=0}^{k-1} (x - x_\mu)$. Therefore $\sigma f'[x_0, x_1, \ldots, x_{k-1}] \geqslant kM$ for every k distinct points in $[a, b]$. A repeated use of this inequality completes the proof of the lemma.

The results in this paper deal with convergence in norms which are generalizations of the L_p-norms $(1 \leqslant p \leqslant \infty)$. Such norms are defined on R_n in [2, p. 40]. Their connection with a similar notion called "Fejér monotonic norm" [3] is discussed in [8].

DEFINITION 1.1. Let K be a set of real functions with domain $[a, b]$, which includes $C[a, b]$ as a subset. A norm $\|\cdot\|$ defined on K is called "monotone" if $f, g \in K$ and $|f(x)| \leqslant |g(x)|$ for $a \leqslant x \leqslant b$ imply $\|f\| \leqslant \|g\|$.

Every monotone norm on $C[a, b]$ is "majorized" by the sup norm $\|\cdot\|_\infty$ in the following sense:

$$\|f\| \leqslant A \|f\|_\infty \qquad \text{for all } f \in C[a, b], \tag{1.5}$$

where A is a constant independent of f. Indeed for every $f(x) \in C[a, b]$

$$|f(x)| \leqslant |l(x)| \cdot \|f\|_\infty, \qquad a \leqslant x \leqslant b,$$

where $l(x) \equiv 1$, and therefore $\|f(x)\| \leqslant \|l\| \cdot \|f\|_\infty$.

We now introduce a useful concept in the investigation of convergence properties of sequences of functions.

DEFINITION 1.2. A sequence $\{g_n(x)\}$ of functions defined on $[a, b]$ is called "nearly convergent" to $g(x)$ on $[a, b]$, if for every subinterval $I \subset [a, b]$ and every $\epsilon > 0$ there exists $N = N(\epsilon)$ such that for every $n > N$ there is an $x_n \in I$ satisfying

$$|g_n(x_n) - g(x_n)| < \epsilon. \tag{1.6}$$

Obviously if $\{g_n(x)\}$ is not nearly convergent to $g(x)$ then there exists a subinterval $I \subset [a, b]$, a subsequence $\{g_{n_j}(x)\}$, and a number $\epsilon > 0$, such that for all j

$$|g_{n_j}(x) - g(x)| \geqslant \epsilon, \text{ for all } x \in I. \tag{1.7}$$

From this we conclude

LEMMA 1.2. *If $\{g_n(x)\}$ converges to $g(x)$ in a monotone norm then it is nearly convergent to $g(x)$ on $[a, b]$.*

Proof. Suppose to the contrary that $\{g_n(x)\}$ is not nearly convergent to $g(x)$ on $[a, b]$. Then for some subinterval $I \subset [a, b]$, $\epsilon > 0$ and a subsequence $\{g_{n_j}\}$, (1.7) holds. Let $f_\epsilon(x) \not\equiv 0$ be a continuous function vanishing on $[a, b] - I$ and satisfying $0 \leqslant f_\epsilon(x) \leqslant \epsilon$ for all $x \in I$. Then $|g_{n_j}(x) - g(x)| \geqslant |f_\epsilon(x)|$, $a \leqslant x \leqslant b$, and thus $\| g_{n_j} - g \| \geqslant \| f_\epsilon \| > 0$, in contradiction to the assumption $\lim_{n\to\infty} \| g_n - g \| = 0$.

2. MAIN RESULTS

We start by proving an auxiliary lemma:

LEMMA 2.1. *Let $I_0, I_1, \dots, I_k$ be disjoint closed subintervals of $[a, b]$. A sequence of functions $\{g_n(x)\}_{n=1}^\infty$ satisfying*

$$|g_n[x_0, x_1, \dots, x_k]| \geqslant \delta > 0 \tag{2.1}$$

for all n and for any $x_j \in I_j$, $j = 0, \dots, k$, is not nearly convergent to zero on $[a, b]$.

Proof. Suppose to the contrary that $\{g_n\}$ is nearly convergent to zero on $[a, b]$. Then for any $\epsilon > 0$ there exists $N = N(\epsilon)$ and $k + 1$ sequences of points $\{x_j^n\}_{n=N}^\infty \subset I_j$, $j = 0, \dots, k$, such that

$$|g_n(x_j^n)| < \epsilon, \qquad j = 0, 1, \dots, k, \quad n > N. \tag{2.2}$$

Let $h = \min\{|x - y| : x \in I_i, y \in I_j, i \neq j\}$, then for $0 < \epsilon < \delta h^k/k$

$$|g_n[x_0^n, x_1^n, ..., x_k^n]| = \left| \sum_{j=0}^{k} \frac{g_n(x_j^n)}{\prod_{\substack{i=0 \\ i\neq j}}^{k} (x_j^n - x_i^n)} \right| < \sum_{j=0}^{k} \frac{\epsilon}{h^k} = \frac{k\epsilon}{h^k} < \delta$$

which contradicts (2.1).

The following theorem characterizes the limit function (in a monotone norm) of a sequence of functions satisfying (for some $k \geqslant 1$):

$$\sigma f_n[x_0, ..., x_k] \geqslant M \qquad \text{for all } a \leqslant x_0 < x_1 < \cdots < x_k \leqslant b, \qquad \sigma = +1 \text{ or } -1. \tag{2.3}$$

THEOREM 2.1. *Let $\{f_n(x)\}_{n=1}^{\infty}$ satisfy* (2.3). *If for some $f \in C[a, b]$*

$$\lim_{n\to\infty} \|f_n - f\| = 0, \tag{2.4}$$

where $\|\cdot\|$ is a monotone norm, then,

$$\sigma f[x_0, ..., x_k] \geqslant M \qquad \textit{for all } a \leqslant x_0 < \cdots < x_k \leqslant b. \tag{2.5}$$

Proof. Suppose to the contrary that there are $a \leqslant y_0 < y_1 < \cdots < y_k \leqslant b$ such that

$$\sigma f[y_0, y_1, ..., y_k] = M - 2\delta, \qquad \delta > 0.$$

Then the continuity of $f(x)$ implies the existence of $k + 1$ disjoint closed subintervals $I_0, I_1, ..., I_k$ of $[a, b]$, such that $y_i \in I_i$, $i = 0, 1, ..., k$, and

$$\sigma f[x_0, x_1, ..., x_k] \leqslant M - \delta \tag{2.6}$$

for any $x_i \in I_i$, $i = 0, 1, ..., k$. Thus for $n \geqslant 1$, $\sigma(f_n - f)[x_0, x_1, ..., x_k] \geqslant \delta > 0$ for all $x_i \in I_i$, $i = 0, 1, ..., k$, and by Lemma 2.1 the sequence $\{f_n - f\}_{n=1}^{\infty}$ is not nearly convergent to zero, which, in view of Lemma 1.2, is in contradiction to (2.4).

LEMMA 2.2. *Let $\{f_n(x)\}_1^{\infty}$ and $f(x)$ be functions satisfying* (2.4).
If for some $j \geqslant 0$, $\{f_n^{(j)}(x)\}_1^{\infty}$ and $f^{(j)}(x)$ exist in a subinterval $I \subset [a, b]$, then the sequence $\{f_n^{(j)}(x)\}$ is nearly convergent to $f^{(j)}(x)$ on I.

Proof. The above result for the case $j = 0$ is proved in Lemma 1.2. To prove it for $j \geqslant 1$, suppose to the contrary that there is a subinterval $I_1 \subset I$ and a subsequence $\{f_{n_k}(x)\}_{k=1}^{\infty}$ such that

$$|f_{n_k}^{(j)}(x) - f^{(j)}(x)| \geqslant \epsilon > 0, \text{ for all } x \in I_1.$$

Then for every $j+1$ distinct points $x_0, x_1, \ldots, x_j$ in I_1 there exists $\xi \in I_1$, such that [13]

$$|(f_{n_k} - f)[x_0, \ldots, x_j]| = \frac{1}{j!}|f_{n_k}^{(j)}(\xi) - f^{(j)}(\xi)| \geqslant \frac{\epsilon}{j!} > 0.$$

Therefore by Lemma 2.1 the sequence $\{f_{n_k} - f\}$ is not nearly convergent to zero on I_1, which is in contradiction to (2.4) in view of Lemma 1.2.

In the rest of this section we investigate the pointwise and uniform convergence of the sequences $\{f_n^{(j)}\}, j \geqslant 0$ for $\{f_n(x)\}$ satisfying (2.3) and (2.4) These results include those of [9] as a special case.

THEOREM 2.2. *Let $\{f_n(x)\}_1^\infty$, f, satisfy* (2.3) *and* (2.4). *If for some $0 \leqslant m < k$, $\{f_n^{(m)}\}$ exist for all n, and $f \in C^m[a, b]$, then for every $x \in (a, b)$*

$$\lim_{n\to\infty} f_n^{(j)}(x) = f^{(j)}(x), j = 0, 1, \ldots, m. \tag{2.7}$$

If, moreover, there exist $\delta > 0$ and $\overline{M}$ such that $\overline{M} > \sup\{\sigma f_n[a, x_1, \ldots, x_k], a < x_1 < \cdots < x_k \leqslant a + \delta, n = 1, 2, \ldots\}$ ($\overline{M} > \sup\{\sigma f_n[b, x_1, \ldots, x_k], b - \delta \leqslant x_1 < \cdots < x_k < b, n = 1, 2, \ldots\}$), then (2.7) *holds for $x = a$ ($x = b$) as well.*

Proof. Suppose there is a point x_0, $a < x_0 < b$, for which (2.7) does not hold. By choosing a subsequence, if necessary, (denoted again by $\{f_n\}$), we assume that for some j, $0 \leqslant j \leqslant m$ either

$$\sigma(f_n^{(j)}(x_0) - f^{(j)}(x_0)) > \epsilon > 0 \text{ for all } n, \tag{2.8}$$

or

$$\sigma(f_n^{(j)}(x_0) - f^{(j)}(x_0)) < -\epsilon < 0 \text{ for all } n. \tag{2.9}$$

By (2.4) and Lemma 2.2 the sequence $f_n^{(j)}(x)$ is nearly convergent to $f^{(j)}(x)$ on $[a, b]$. Let $\delta > 0$ be such that for every $x, y \in [x_0 - \delta, x_0 + \delta] \subset [a, b]$

$$|f^{(j)}(x) - f^{(j)}(y)| < \frac{\epsilon}{[4(k-j)]^{k-j+1}} \tag{2.10}$$

and let $[x_0 - \delta, x_0 + \delta]$ be divided into $4(k-j)$ subintervals of length $\delta/2(k-j)$ each. Since $\{f_n^{(j)}\}$ is nearly convergent to $f^{(j)}$, there exists N such that for all $n > N$ it is possible to choose points $x_1^n, \ldots, x_{k-j}^n$ in some $k-j$ of the above intervals, for which:

$$\frac{\delta}{2(k-j)} \leqslant |x_\nu^n - x_\mu^n| \leqslant 2\delta, \nu \neq \mu, \mu, \nu = 0, \ldots, k-j, x_0^n \equiv x_0, \quad n \geqslant 1, \tag{2.11}$$

$$|f_n^{(j)}(x_\nu^n) - f^{(j)}(x_\nu^n)| < \frac{\epsilon}{[4(k-j)]^{k-j+1}}, \nu = 1, \ldots, k-j, \tag{2.12}$$

and

$$\frac{\sigma[f_n^{(j)}(x_0) - f^{(j)}(x_0)]}{\prod_{\nu=1}^{k-j}(x_0{}^n - x_\nu{}^n)} < 0. \tag{2.13}$$

Now

$$\begin{aligned}\sigma f_n^{(j)}[x_0, x_1{}^n, \ldots, x_{k-j}^n] &= \sigma(f_n^{(j)} - f^{(j)}(x_0))[x_0, x_1{}^n, \ldots, x_{k-j}^n] \\ &= \sum_{\nu=0}^{k-j} \frac{\sigma[f_n^{(j)}(x_\nu{}^n) - f^{(j)}(x_0)]}{\Pi_\nu},\end{aligned}$$

where

$$\Pi_\nu = \prod_{\substack{\mu=0\\ \mu\neq\nu}}^{k-j} (x_\nu{}^n - x_\mu{}^n),\ \nu = 0, 1, \ldots, k-j.$$

For $\nu = 1, 2, \ldots, k-j$ we have by (2.10)–(2.12)

$$\begin{aligned}&\left|\frac{\sigma[f_n^{(j)}(x_\nu{}^n) - f^{(j)}(x_0)]}{\Pi_\nu}\right| \\ &\quad\leqslant \frac{|f_n^{(j)}(x_\nu{}^n) - f^{(j)}(x_\nu{}^n)| + |f^{(j)}(x_\nu{}^n) - f^{(j)}(x_0)|}{|\Pi_\nu|} \\ &\quad\leqslant \frac{2\epsilon}{[4(k-j)]^{k-j+1}} \cdot \left(\frac{\delta}{2(k-j)}\right)^{j-k} = \frac{\epsilon}{2(k-j)(2\delta)^{k-j}}.\end{aligned} \tag{2.14}$$

Thus the sum of the last $k-j$ terms in (2.14) does not exceed in absolute value $\frac{1}{2}(\epsilon/(2\delta)^{k-j})$. On the other hand, by (2.8) or (2.9) and by (2.11) and (2.13)

$$\frac{\sigma[f_n^{(j)}(x_0) - f^{(j)}(x_0)]}{\Pi_0} < -\frac{\epsilon}{(2\delta)^{k-j}}.$$

Therefore

$$\sigma f_n^{(j)}[x_0{}^n, \ldots, x_{k-j}^n] < -\frac{\epsilon}{2(2\delta)^{k-j}}, \tag{2.15}$$

which in view of Lemma 1.1, for $\delta > 0$ small enough contradicts (2.3).

Suppose now that (2.7) does not hold at $x_0 = a$ ($x_0 = b$) for some $0 \leqslant j \leqslant m$. Then as in (2.8) and (2.9) we have

$$|f_n^{(j)}(x_0) - f^{(j)}(x_0)| > \epsilon, \qquad x_0 = a\ (x_0 = b).$$

Although in the interval $[a, a+\delta]$ ($[b-\delta, b]$), $\delta > 0$, (2.13) may not hold, yet the same calculations leading to (2.15) yield the weaker inequality

$$|f_n^{(j)}[x_0, x_1{}^n, \ldots, x_{k-j}^n]| > \frac{\epsilon}{2(2\delta)^{k-j}}, \qquad x_0 = a\ (x_0 = b). \tag{2.16}$$

By Lemma 1.1, since δ can be arbitrarily small, (2.16) is consistent with (2.3) only if

$$\sigma f_n^{(j)}[x_0, x_1^n, ..., x_{k-j}^n] > \frac{\epsilon}{2(2\delta)^{k-j}}, \qquad x_0 = a \qquad (x_0 = b), \tag{2.17}$$

which excludes the existence of a bound $\bar{M}$ such that $\sigma f_n[x_0, x_1, ..., x_k] < \bar{M}$ $x_0 = a(x_0 = b)$ for all distinct $x_1, ..., x_k$ in $[a, a + \delta]$ ($[b - \delta, b]$). This completes the proof of the theorem.

Notice that, since for $x_0 = a$, $\operatorname{sgn}[\prod_{\nu=1}^{k-j} (x_0 - x_\nu^n)] = (-1)^{k-j}$ and for $x_0 = b$, $\prod_{\nu=1}^{k-j} (x_0 - x_\nu^n) > 0$, it follows from the calculations leading to (2.17) that if there is no convergence, necessarily

$$(-1)^{k-j} \sigma\{f_n^{(j)}(a) - f^{(j)}(a)\} > 0,$$
$$\sigma\{f_n^{(j)}(b) - f^{(j)}(b)\} > 0.$$

The following theorem relates pointwise convergence of sequences satisfying (2.3) with their uniform convergence.

THEOREM 2.3. *Let $\{f_n(x)\}_1^\infty$ satisfy*

$$\sigma f_n[x_0, x_1, ..., x_k] \geqslant M \tag{2.18}$$

for some $k \geqslant 1$, for all n, and for all $\alpha \leqslant x_0 < x_1 < \cdots x_k \leqslant \beta$, where $\sigma = +1$ or -1. If for some $f \in C[\alpha, \beta]$

$$\lim_{n \to \infty} f_n(x) = f(x) \text{ for every } \alpha \leqslant x \leqslant \beta, \tag{2.19}$$

then $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$, where $\|\cdot\|_\infty$ is the sup-*norm over $[\alpha, \beta]$.*

Proof. Suppose to the contrary that $\lim_{n\to\infty} \|f_n - f\|_\infty \neq 0$. Then there exist $\epsilon > 0$, a subsequence (denoted again by $\{f_n\}$), and a sequence $\{x_n\} \subset [\alpha, \beta]$ for which

$$|f_n(x_n) - f(x_n)| = \|f_n - f\|_\infty > \epsilon. \tag{2.20}$$

Let us treat the case

$$\sigma(f_n(x_n) - f(x_n)) > \epsilon,$$
$$x_n > x_0, \lim_{n\to\infty} x_n = x_0, \alpha \leqslant x_0 < \beta.$$

All other possibilities can be treated similarly as can be seen from the proof.

Let $\delta > 0$ be such that for every $x, y \in [x_0, x_0 + \delta]$

$$|f(x) - f(y)| < \epsilon/2^{k+3}k^k. \tag{2.21}$$

Let $\xi_1, \xi_2, ..., \xi_k$ be k fixed points in $[x_0 + \delta/2, x_0 + \delta]$ for which $|\xi_\nu - \xi_\mu| \geqslant \delta/2k$, $\nu \neq \mu$.

By (2.19) there exists N such that for every $n > N$

$$|f_n(x_0) - f(x_0)| < \frac{\epsilon}{2^{k+3}k^k}, \tag{2.22}$$

$$|f_n(\xi_\nu) - f(\xi_\nu)| < \frac{\epsilon}{2^{k+3}k^k}, \qquad \nu = 1, 2,..., k, \tag{2.23}$$

$$x_0 \leqslant x_n \leqslant x_0 + \delta/4. \tag{2.24}$$

For k odd, $\prod_{\nu=1}^{k} (x_n - \xi_\nu) < 0$ and

$$\begin{aligned}
&\sigma f_n[x_n, \xi_1, \xi_2, ..., \xi_k] \\
&\quad = \sigma(f_n - f(x_n))[x_n, \xi_1, ..., \xi_k] \\
&\quad = \frac{\sigma(f_n(x_n) - f(x_n))}{\prod_{\nu=1}^{k} (x_n - \xi_\nu)} + \sum_{\nu=1}^{k} \frac{\sigma(f_n(\xi_\nu) - f(\xi_\nu)) + \sigma(f(\xi_\nu) - f(x_n))}{\prod_{\substack{\mu=1 \\ \mu\neq\nu}}^{k} (\xi_\nu - \xi_\mu)(\xi_\nu - x_n)} \\
&\quad \leqslant -\frac{\epsilon}{\delta^k} + k\frac{2(\epsilon/2^{k+3}k^k)}{(\delta/2k)^{k-1}(\delta/4)} = -\frac{1}{2}\frac{\epsilon}{\delta^k},
\end{aligned}$$

which can be made infinitely negative as $\delta \to 0$, in contradiction to (2.18).

For k even, $(x_n - x_0)\prod_{\nu=1}^{k-1} (x_n - \xi_\nu) < 0$ and the contradiction is achieved by showing that $\sigma f_n[x_0, x_n, \xi_1, ..., \xi_{k-1}]$ does not satisfy (2.18):

$$\begin{aligned}
&\sigma f_n[x_0, x_n, \xi_1, ..., \xi_{k-1}] \\
&\quad = \frac{\sigma(f_n(x_n) - f(x_n))}{(x_n - x_0)\prod_{\nu=1}^{k-1} (x_n - \xi_\nu)} + \frac{\sigma(f_n(x_0) - f(x_0)) + \sigma(f(x_0) - f(x_n))}{(x_0 - x_n)\prod_{\nu=1}^{k-1} (x_0 - \xi_\nu)} \\
&\qquad + \sum_{\nu=1}^{k-1} \frac{\sigma(f_n(\xi_\nu) - f(\xi_\nu)) + \sigma(f(\xi_\nu) - f(x_n))}{(\xi_\nu - x_0)(\xi_\nu - x_n)\prod_{\substack{\mu=1 \\ \mu\neq\nu}}^{k-1} (\xi_\nu - \xi_\mu)} \leqslant -\frac{1}{2}\frac{\epsilon}{\delta^k},
\end{aligned}$$

where the last inequality is derived as in the odd case.

As a consequence of the last results we have

COROLLARY 2.1. *Under the assumptions of Theorem* 2.2

$$\lim_{n\to\infty} \|f_n^{(j)} - f^{(j)}\|_\infty = 0, \qquad j = 0, 1,..., m,$$

where $\|\cdot\|_\infty$ *is the* sup-*norm over* $[\alpha, \beta]$, $a < \alpha < \beta < b$.

This result was proved in [9] for the case $j = 0$ and the L_p-norms $1 \leqslant p < \infty$.

COROLLARY 2.2. *If in Theorem* 2.2 *assumption* (2.3) *is replaced by the stronger assumption*

$$L \leqslant f_n[x_0, x_1, ..., x_k] \leqslant U \qquad \textit{for all } n \tag{2.25}$$

and for every $a \leqslant x_0 < x_1 < \cdots < x_k \leqslant b$, *where* L *and* U *are constants, then*

$$\lim_{n\to\infty} \| f_n^{(j)} - f^{(j)} \|_\infty = 0, \qquad j = 0, 1,..., m, \tag{2.26}$$

where $\| \cdot \|_\infty$ *is the* sup *norm over* $[a, b]$.

Proof. By (2.25) and the last assertion of Theorem 2.2 it follows that

$$\lim_{n\to\infty} f_n^{(j)}(x) = f^{(j)}(x), \qquad a \leqslant x \leqslant b, j = 0, 1,..., m.$$

This together with Theorem 2.3 yields (2.26).

3. Applications to "Restricted Derivative" Approximation

In this section the results of Section 2 are applied to the evaluation of the degree of approximation in "Restricted Derivative" approximation (R.D.A.) and "Monotone Approximation" (M.A.) [10, 14].

The problem called R.D.A. deals with approximation of functions by polynomials from the class

$$K_n = \{p \mid p \in \Pi_{n-1}, l_i(x) \leqslant p^{(k_i)}(x) \leqslant u_i(x), a \leqslant x \leqslant b, i = 0, 1,..., s\}, \tag{3.1}$$

where $\prod_{n-1}$ is the class of all polynomials of degree $\leqslant n - 1$, $0 \leqslant k_0 < k_1 < \cdots < k_s \leqslant n - 1$, $l_i(x) < u_i(x)$, $a \leqslant x \leqslant b$, $i = 0, 1,..., s$ and $\{l_i(x)\}_{i=0}^s$ $[\{u_i(x)\}_{i=0}^s]$ may take the value $-\infty$ $[+\infty]$ on open subsets of $[a, b]$ and are continuous elsewhere in $[a, b]$. Moreover we assume that there exists $h \in C^{k_s}[a, b]$ for which $l_i(x) < h^{(k_i)}(x) < u_i(x)$, $a \leqslant x \leqslant b$, $i = 0, 1,..., s$.

Monotone Approximation is a special case of R.D.A. where the class of approximating polynomials is

$$M_n = \{p \mid p \in \Pi_{n-1}, \epsilon_i p^{(k_i)}(x) \geqslant 0, a \leqslant x \leqslant b, i = 1, 2,..., s\}, \tag{3.2}$$

where $1 \leqslant k_1 < k_2 < \cdots < k_s \leqslant n - 1$, $\epsilon_i = +1$ or -1.

Existence, uniqueness, and characterizations of the polynomial of best approximation (p.b.a.) from K_n and M_n to a given function in the sup-norm $\| \cdot \|_\infty$, are treated in the literature [6, 10, 14]. On the other hand, estimations of the degree of approximation from K_n as $n \to \infty$ with fixed constraints are not yet known. The only results in this direction deal with the special case of M.A., the sup-norm and a single constraint ($s = 1$ in (3.2)) [4, 12, 15].

Let $\| \cdot \|$ be a monotone norm defined on $C[a, b]$. Denote by

$$E(f, \Phi) = \inf_{p\in\Phi} \| f - p \| \tag{3.3}$$

the degree of approximation of a given function by functions from a class Φ.

It is easily seen that for functions satisfying $f \in C^{k_s}[a, b]$ and $l_i(x) \leqslant f^{(k_i)}(x) \leqslant u_i(x)$, $a \leqslant x \leqslant b$, $i = 0, 1, \ldots, s$:

$$\lim_{n\to\infty} E(f, K_n) = 0. \tag{3.4}$$

To verify this, let $\epsilon > 0$ be fixed. Since there exists a function $h \in C^{k_s}[a, b]$ for which $l_i(x) < h^{(k_i)}(x) < u_i(x)$, $a \leqslant x \leqslant b$, $i = 0, 1, \ldots, s$, then any function $g_\lambda = (1 - \lambda) f + \lambda h$, $0 < \lambda < 1$, satisfies the same strict inequalities. Moreover for λ small enough $\| f - g_\lambda \|_\infty = \lambda \| f - h \|_\infty < \epsilon/2$. By [11] there exists a polynomial $p_n \in \Pi_{n-1}$ for which $\max_{i=0,1,\ldots,k_s} \| g_\lambda^{(i)} - p_n^{(i)} \|_\infty < \min_{a\leqslant x\leqslant b, i=0,1,\ldots,s}\{\epsilon/2, g_\lambda^{(i)}(x) - l_i(x), u_i(x) - g_\lambda^{(i)}(x)\}$.

Therefore $p_n \in K_n$ and $\| f - p_n \|_\infty \leqslant \| f - g_\lambda \|_\infty + \| g_\lambda - p_n \|_\infty < \epsilon$, which together with (1.5) proves (3.4).

Using the results of Section 2 and under certain assumptions on the approximated function $f(x)$ and the ranges $\{l_i(x)\}_0^s$, $\{u_i(x)\}_0^s$, we hereby prove that in order to estimate $E(f, K_n)$ it is enough to consider the degree of approximation of f by polynomials with only one restricted derivative. A similar, but weaker result holds for the case of M.A.

THEOREM 3.1. *Let K_n be defined by* (3.1), *let*

$$\bar{K}_n = \{ p \mid p \in \Pi_{n-1}, l_s(x) \leqslant p^{(k_s)}(x) \leqslant u_s(x), a \leqslant x \leqslant b\}, \tag{3.5}$$

and suppose $l_s(x)$ and $u_s(x)$ are bounded on $[a, b]$. Then, for $f(x) \in C^{k_s}[a, b]$ satisfying

$$\begin{aligned} l_i(x) &< f^{(k_i)}(x) < u_i(x), \qquad a \leqslant x \leqslant b,\ i = 0, 1, \ldots, s - 1 \\ l_s(x) &\leqslant f^{(k_s)}(x) \leqslant u_s(x), \qquad a \leqslant x \leqslant b, \end{aligned} \tag{3.6}$$

there exists N such that

$$E(f, K_n) = E(f, \bar{K}_n), \qquad n \geqslant N. \tag{3.7}$$

Proof. Let $\bar{p}_n$ be a p.b.a. to f from $\bar{K}_n$. Since $K_n \subset \bar{K}_n$, $E(f, \bar{K}_n) \leqslant E(f, K_n)$, and thus to establish (3.7) it is enough to show that for $n \geqslant N$, $\bar{p}_n \in K_n$.

By (3.4), $\lim_{n\to\infty} \| f - \bar{p}_n \| = 0$, and since $l_s(x)$ and $u_s(x)$ are bounded

$$\| \bar{p}_n^{(k_s)} \|_\infty \leqslant \max\{\| l_s \|_\infty, \| u_s \|_\infty\} = M, \tag{3.8}$$

$$\| f^{(k_s)} \|_\infty \leqslant M. \tag{3.9}$$

But for each $a \leqslant x_0 < x_1 < \cdots < x_{k_s} \leqslant b$ there exists $\xi \in (a, b)$ such that $\bar{p}_n[x_0, x_1, ..., x_{k_s}] = \bar{p}_n^{(k_s)}(\xi)/k_s!$ [13], and therefore the sequence $\{\bar{p}_n\}$ and $f(x)$ satisfy the conditions of Corollary 2.2, from which we conclude that

$$\lim_{n\to\infty} \| f^{(i)} - \bar{p}_n^{(i)} \|_\infty = 0, \qquad i = 0, 1, ..., k_s - 1. \tag{3.10}$$

Now (3.6) together with (3.10) implies the existence of N such that for all $n > N$

$$l_i(x) < \bar{p}_n^{(k_i)}(x) < u_i(x), \qquad a \leqslant x \leqslant b, i = 0, 1, ..., s-1,$$

and thus $\bar{p}_n \in K_n$, $n > N$.

The last result does not hold for M.A. since either $l_s(x)$ or $u_s(x)$ is unbounded. Yet, Theorem 2.2 enables us to prove something similar for this case. To this end, let us associate with every monotone norm $\| \cdot \|$ on $C[a, b]$ a monotone norm $\| \cdot \|_\delta$ on $C[a - \delta, b + \delta]$, $\delta > 0$, such that

$$\|f\| \leqslant \|f\|_\delta, \qquad f \in C[a - \delta, b + \delta]. \tag{3.11}$$

Obviously such monotone norms exist; for example, we may choose

$$\|f\|_\delta = \|f\| + \sup_{\substack{a-\delta \leqslant x \leqslant a \\ b \leqslant x \leqslant b+\delta}} |f(x)|.$$

THEOREM 3.2. *Let M_n be defined by* (3.2), *and for any $\delta \geqslant 0$ let*

$$\overline{M}_n^{\delta} = \{ p \mid p \in \Pi_{n-1}, \epsilon_s p^{(k_s)}(x) \geqslant 0, a - \delta \leqslant x \leqslant b + \delta \}. \tag{3.12}$$

Then for $f(x)$ satisfying

$$\begin{aligned} &f(x) \in C^{k_s}[a - \delta, b + \delta], \\ &\epsilon_s f^{(k_s)}(x) \geqslant 0, \qquad a - \delta \leqslant x \leqslant b + \delta, \\ &\epsilon_i f^{(k_i)}(x) > 0, \qquad i = 1, 2, ..., s-1, a \leqslant x \leqslant b, \end{aligned} \tag{3.13}$$

there exists $N = N(\delta)$ such that

$$E(f, \overline{M}_n^0) \leqslant E(f, M_n) \leqslant E_\delta(f, \overline{M}_n^\delta), \qquad n \geqslant N(\delta), \tag{3.14}$$

where

$$E_\delta(f, \overline{M}_n^\delta) = \inf_{p_n \in \overline{M}_n^\delta} \|f - p_n\|_\delta$$

and $\| \cdot \|_\delta$ is a monotone norm on $C[a - \delta, b + \delta]$ satisfying (3.11).

Proof. Since $M_n \subset \overline{M}_n^0$, the left-hand-side inequality in (3.14) is obvious. Denote by $\bar{p}_n$ a p.b.a. to f from $\overline{M}_n^\delta$ with respect to the monotone norm $\| \cdot \|_\delta$. Since by (3.4), $\lim_{n\to\infty} \|f - \bar{p}_n\|_\delta = 0$, it follows from Corollary 2.1

that $\lim_{n\to\infty} \| f^{(i)} - \bar{p}_n^{(i)} \|_\infty = 0$, $i = 0, 1, \ldots, k_s - 1$, where $\| \cdot \|_\infty$ is the sup-norm on $[a, b]$.
Thus, in view of (3.13), there exists $N(\delta)$ such that $\bar{p}_n \in M_n$, $n \geqslant N(\delta)$. This shows that

$$E(f, M_n) \leqslant \| f - \bar{p}_n \|, \qquad n \geqslant N(\delta).$$

By (3.11)

$$\| f - \bar{p}_n \| \leqslant \| f - \bar{p}_n \|_\delta = E_\delta(f, \bar{M}_n{}^\delta),$$

which completes the proof of the theorem.

Theorems 3.1 and 3.2 indicate the importance of investigating the degree of approximation in M.A. and R.D.A. for the special case $s = 1$, $k_1 > 1$. The only bound known for this case [15] is not the best possible. Moreover, there is no point in applying it to the estimation of $E(f, M_n)$ via Theorem 3.2, since the method of repeated integration used in [15] can be modified to give a similar bound without the requirement of strict inequalities in (3.13).

More specifically, let q be the p.b.a. in the sup-norm from Π_{n-1-k_s} to $f^{(k_s)}$ and let $E = \| f^{(k_s)} - q \|_\infty$. Then for f satisfying $\epsilon_i f^{(k_i)}(x) \geqslant 0$, $a \leqslant x \leqslant b$, $i = 1, \ldots, s$, since

$$\operatorname{sgn} [f^{(k_s)} - q - \epsilon_s E] = -\operatorname{sgn} \epsilon_s,$$

the polynomial $p_{n-1-k_s} = q + \epsilon_s E$ satisfies

$$\epsilon_s (f^{(k_s)} - p_{n-1-k_s}) \leqslant 0,$$

and therefore

$$\epsilon_s p_{n-1-k_s}(x) \geqslant \epsilon_s f^{(k_s)}(x) \geqslant 0, \qquad a \leqslant x \leqslant b.$$

By defining $(f^{(k_s-1)} - p_{n-k_s})$ either as

$$\int_a^x (f^{(k_s)}(t) - p_{n-1-k_s}(t))\, dt \qquad \text{or as} \qquad \int_b^x (f^{(k_s)}(t) - p_{n-1-k_s}(t))\, dt$$

we can choose its sign on $[a, b]$. Repeating this process k_s times we construct a polynomial $p_{n-1} \in \Pi_{n-1}$ which satisfies:

$$\epsilon_j (f(x) - p_{n-1}(x))^{(k_j)} \leqslant 0 \text{ and therefore } \epsilon_j p_{n-1}^{(k_j)}(x) \geqslant \epsilon_j f^{(k_j)}(x) \geqslant 0$$

for $x \in [a, b]$, $j = 1, 2, \ldots, s$. By this construction $p_{n-1} \in M_n$ and by (1.5)

$$\begin{aligned} E(f, M_n) &\leqslant \| f - p_{n-1} \| \leqslant A \| f - p_{n-1} \|_\infty \\ &\leqslant 2A(b - a)^{k_s} \inf_{p \in \Pi_{n-1-k_s}} \| f^{(k_s)} - p \|_\infty . \end{aligned}$$

Since in the case of M.A. with $s = 1$, $k_1 = 1$ a best possible bound for the degree of approximation is known [4, 12], the above method yields a somewhat better estimate for $E(f, M_n)$. Indeed, starting with the p.b.a. to $f^{(k_s-1)}$ in the sup-norm for the case $k_1 = 1$, $s = 1$, and applying the method of repeated integration, we get

$$E(f, M_n) \leqslant 2A(b-a)^{k_s-1} \inf\{\| f^{(k_s-1)} - p \|_\infty, p \in \Pi_{n-k_s}, \\ \epsilon_s p'(x) \geqslant 0, a \leqslant x \leqslant b\}.$$

In view of the results in [4], we have for functions $f \in C^r[a, b]$, $r \geqslant k_s$.

$$E(f, M_n) \leqslant \frac{2C(b-a)^{k_s-1}}{(n-k_s)^{r-k_s+1}} \omega\left(f^{(r)}, \frac{b-a}{n-k_s}\right), \qquad n \geqslant r + k_s + 1,$$

where C is a constant depending only on r.

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